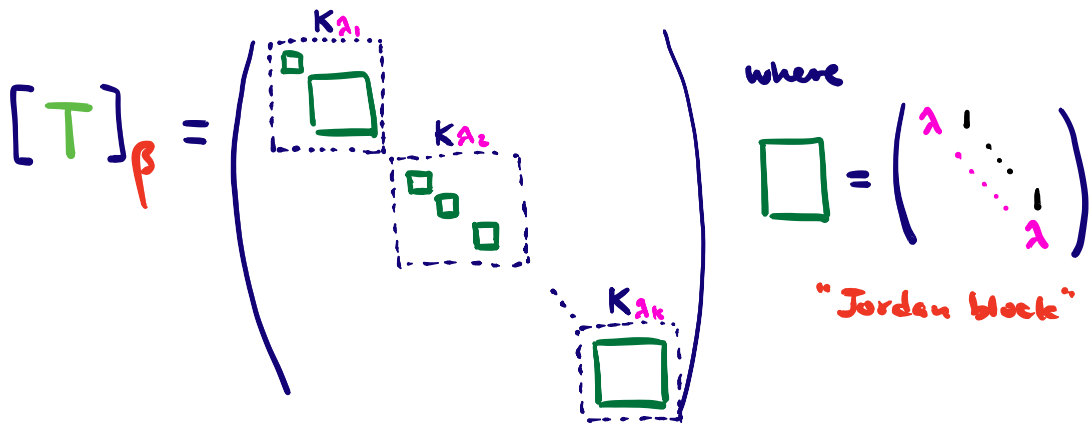
MATH 2040 Lecture 22 (28/11/2016)

Recall: $T : V \longrightarrow V$, assume $\mathbb{F} = \mathbb{C}$, $\dim V < +\infty$.

Find a "Jordan canonical basis" β of V s.t.



Last time: $V = K_{\lambda_1} \oplus \cdots \oplus K_{\lambda_k}$

* **Digression:** (You should be able to do the following:)

(1) Find a Jordan canonical basis and a Jordan canonical form for an operator $T : V \to V$.

(2) Diagonalize a symmetric (real) matrix by an orthonormal eigenbasis.

(3) Given $T : V \to V$ normal, find a polynomial g s.t.

$$T^* = g(T).$$

Recall:

$$\begin{array}{ccccccc} V & = & K_{\lambda_1} & \oplus \cdots \oplus & K_{\lambda_k} & & \\ \cup & & \cup & & \cup & & \\ \beta & = & \beta_1 & \cup \cdots \cup & \beta_k & & \text{basis} \end{array}$$

$\Longrightarrow$

Q: How to find a "good" basis β_i for K_{λ_i} ?

Recall: $A = \begin{pmatrix} \lambda & 1 & \\ & \lambda & 1 \\ & & \lambda \end{pmatrix}$ (e_1)

std basis = $\{e_1, e_2, e_3\}$

$$\Rightarrow \begin{cases} (A - \lambda I)e_1 = 0 \\ (A - \lambda I)e_2 = e_1 \\ (A - \lambda I)e_3 = e_2 \end{cases}$$

$$\beta = \{e_1, e_2, e_3\} = \{\underbrace{(A-\lambda I)^2 e_3, (A - \lambda I)e_3, e_3}_{\text{cycle}}\}$$

Def^{n}: Let $\overset{\neq}{x}$ ($\neq 0$) $x \in K_\lambda$, and $p \geq 1$ be the smallest integer

s.t. $(T - \lambda I)^p x = 0$.

Then, $\{\underbrace{(T-\lambda I)^{p-1}x}_{\text{initial vector}}, (T-\lambda I)^{p-2}x, \ldots, \underbrace{x}_{\text{end vector}}\}$ "cycle"

Remark: • $p \leq$ multi. of λ

• initial vector $\in E_\lambda$

Lemma: Each K_λ has a basis consisting of unions of cycles.

Pf: next time.

Example 1 :

$$A = \begin{pmatrix} -1 & -1 & 0 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{pmatrix}$$

Find Jordan canonical form & basis.

Sol : Step 1 : Find eigenvalues

A upper triangular $\Rightarrow$ $\boxed{\lambda = -1}$ only 1 e.value

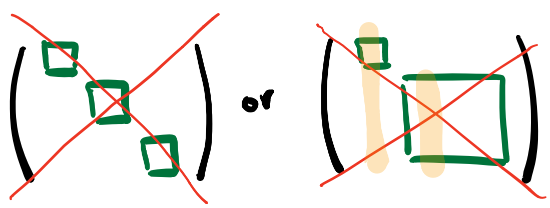
$\therefore \; V = \mathbb{C}^3 = K_{-1}$

Step 2 : Find the eigenspace E_λ.

$$E_\lambda = N(A - \lambda I) = N\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\} \qquad \boxed{\dim E_{-1} = 1 < 3}$$

Step 3 : Determine Jordan canonical form.

$J =$ 

$\because$ not diagonalizable

otherwise $\dim E_{-1} = 2$

or $\begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}$ ← this one

Step 4 : Find Jordan canonical basis.

$$\beta = \{ (A - \lambda I)^2 v, \; (A - \lambda I) v, \; v \}$$

← find this! (v)

Choose $v \in K_{-1} = \mathbb{C}^3$ s.t.

$$v \notin N((A - \lambda I)^2)$$

Compute: $A + I = \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$

$$(A+I)^2 = \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow N(A+I)^2 = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

e.g. Take $v = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, then

$$(A+I)v = \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}$$

$$(A+I)^2 v = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \beta = \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$ is Jordan can. basis

Example 2

$$A = \begin{pmatrix} 2 & -1 & 0 & 1 \\ 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -1 & 0 & 3 \end{pmatrix} \in M_{4\times 4}(\mathbb{C})$$

Sol^{n}: **Step 1: Eigenvalues**

$$f(t) = \det(A - tI)$$

$$= \det \begin{pmatrix} 2-t & -1 & 0 & 1 \\ 0 & 3-t & -1 & 0 \\ 0 & 1 & 1-t & 0 \\ 0 & -1 & 0 & 3-t \end{pmatrix}$$

$$= (2-t) \det \begin{pmatrix} 3-t & -1 & 0 \\ 1 & 1-t & 0 \\ -1 & 0 & 3-t \end{pmatrix}$$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & \end{pmatrix} \ldots$$

$$= (2-t)(3-t) \det \begin{pmatrix} 3-t & -1 \\ 1 & 1-t \end{pmatrix}$$

$$= (t-2)(t-3)\left[\underbrace{(3-t)(1-t)+1}_{t^2-4t+4}\right]$$

$$f(t) = (t-2)^3(t-3)$$

$$\Rightarrow \boxed{\begin{array}{ll} \lambda_1 = 2, & m_1 = 3 \\ \lambda_2 = 3, & m_2 = 1 \end{array}}$$

Step 2: Eigenspace.

$$E_{\lambda_1} = N(A - 2I) = N\begin{pmatrix} 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} = \overset{\dim = 2}{\text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\}}$$

$$E_{\lambda_2} = N(A-3I) = N\begin{pmatrix} -1 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}\right\} \quad \dim = 1$$

Step 3: Determine Jordan can. form.

$$J = \begin{pmatrix} \boxed{2} & & & \\ & \boxed{2} & & \\ & & \boxed{2} & \\ & & & \boxed{3} \end{pmatrix} \text{ or } \begin{pmatrix} \boxed{2} & & & \\ & 2 & 1 & \\ & & 2 & \\ & & & \boxed{3} \end{pmatrix} \text{ or } \begin{pmatrix} 2 & 1 & & \\ & 2 & 1 & \\ & & 2 & \\ & & & \boxed{3} \end{pmatrix}$$

↑ this is it! (the middle one)

$\because \dim E_2 = 2$

[**Note:** # of $\square$'s for λ = $\dim E_\lambda$]

Step 4: Find Jordan can. basis.

$$\beta = \{v_1\} \cup \{(A-2I)v_2, v_2\} \cup \{v_3\}, \quad v_3 \in E_3$$

$\boxed{2}$ $\boxed{\begin{matrix} 2 & 1 \\ & 2 \end{matrix}}$ $\boxed{3}$

- $v_3 \in E_3$ say $v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

- For $v_2 \in K_2$ and $v_2 \notin N(A-2I) = E_2$

$$K_2 = N(A-2I)^2$$

$$= N\begin{pmatrix} 0 & -2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 1 & 1 \end{pmatrix} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 2 \end{pmatrix}\right\}$$

Take $v_2 = \begin{pmatrix} 0 \\ 1 \\ 2 \\ 0 \end{pmatrix} \in K_2 \setminus E_2$.

- Last, $v_1 \in E_2$ not dependent w/ $(A - 2I)v_2 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \end{pmatrix}$

  e.g. $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

$\square$

Example 3:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Eigenvalue: $\lambda = 0$, $m = 4$ ($\because$ upper triangular)

Eigenspace: $E_0 = N(A) = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

dim $= 2$

$$J = \underbrace{\begin{pmatrix} 0 & & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}}_{J_1} \quad \text{or} \quad \underbrace{\begin{pmatrix} 0 & 1 & & \\ & 0 & & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix}}_{J_2}$$

Note:

$$\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 1 & \\ & \ddots & \ddots & \ddots \\ & & \ddots & 0 \\ & & & 0 \end{pmatrix}$$

$J_1^3 = 0$, $J_1^2 \neq 0$

$J_2^2 = 0$

$Q^{-1}AQ = J \implies$ Q: $A^2 = 0$?

Check: $A^2 = \begin{pmatrix} 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0. \Rightarrow J = J_1$

$\beta = \{v_1\} \cup \{A^2 v_2, A v_2, v_2\}$ Exercise!

Find these!